

ON NONCOMMUTATIVE BASES OF THE FREE MODULE $W_n(\mathbb{K})$ OF ALL \mathbb{K} -DERIVATIONS OF THE POLYNOMIAL RING IN n VARIABLES

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ABSTRACT. Let \mathbb{K} be an algebraically closed field of characteristic zero and $R = \mathbb{K}[x_1, x_2, \dots, x_n]$ the polynomial ring in n variables over \mathbb{K} . We study bases of the free R -module $W_n(\mathbb{K})$ of all \mathbb{K} -derivations of the ring R , such that their linear span over \mathbb{K} is a subalgebra of the Lie algebra $W_n(\mathbb{K})$. We proved that for any Lie algebra L of dimension n over \mathbb{K} there exists a subalgebra \overline{L} of $W_n(\mathbb{K})$ which is isomorphic to L and such that every \mathbb{K} -basis of \overline{L} is an R -basis of the R -module $W_n(\mathbb{K})$.

1. INTRODUCTION

Let \mathbb{K} be an algebraically closed field of characteristic 0 and $W_n(\mathbb{K})$ the Lie algebra of all \mathbb{K} -derivations of the polynomial $R = \mathbb{K}[x_1, x_2, \dots, x_n]$ in n variables over \mathbb{K} . The structure of the Lie algebra $W_n(\mathbb{K})$ and its subalgebras was studied by many authors (see, for example, [1, 3, 7]). One of the most important problem here is the question about structure of finite dimensional subalgebras of $W_n(\mathbb{K})$. The description of all such subalgebras in $W_1(\mathbb{K})$ and $W_2(\mathbb{K})$ in case of the field $\mathbb{K} = \mathbb{C}$ of complex numbers can be easily obtained from the works of S.Lie [4]. There is no such a description for the Lie algebra $W_n(\mathbb{K})$, $n \geq 3$, so it is of interest to study some classes of finite dimensional subalgebras of the algebra $W_n(\mathbb{K})$.

Define one of such classes in the following way: a subalgebra L of $W_n(\mathbb{K})$ of dimension n over \mathbb{K} will be called *basic*, if every basis of the algebra L over \mathbb{K} is a basis of the R -module $W_n(\mathbb{K})$. It is obvious, that the Lie algebra $\mathbb{K}\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \rangle$ is abelian and basic. All abelian basic Lie subalgebras in $W_n(\mathbb{K})$ are described in [5] (see also [6]): let $f_1, f_2, \dots, f_n \in R$ such that $\det J(f_1, f_2, \dots, f_n) \in \mathbb{K}^*$, where $J(f_1, f_2, \dots, f_n)$ is the Jacoby matrix of polynomials $f_1, f_2, \dots, f_n \in R$. Then the derivations $D_1, \dots, D_n \in W_n(\mathbb{K})$ defined by the conditions $D_i(h) = \det J(f_1, \dots, f_{i-1}, h, f_{i+1}, \dots, f_n)$ for any $h \in R$ pairwise commute and form a basis of the R -module $W_n(\mathbb{K})$. Conversely, every basis $\{D_1, D_2, \dots, D_n\}$ such that $[D_i, D_j] = 0$, $i, j = 1, \dots, n$, can be obtained in such a way. But there exist also non-abelian basic subalgebras. For example, the two-dimensional subalgebra with a basis $\{\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\}$ from $W_2(\mathbb{K})$ is basic and non-abelian. The main result of the paper, Theorem 2, shows that every Lie algebra of dimension n over \mathbb{K} is isomorphic to a basic subalgebra in $W_n(\mathbb{K})$. In Theorem 1, all nilpotent basic subalgebras of the Lie algebra $W_n(\mathbb{K})$ are characterized.

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We use standard notation in the paper (see, for example, [6]). The ground field \mathbb{K} is algebraically closed of characteristic 0. The basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$ of the free R -module $W_n(\mathbb{K})$ will be called standard. Every another basis $\{D_1, D_2, \dots, D_n\}$ can be obtained from the standard one by an invertible

$$\text{matrix } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \text{ where } D_i = a_{i1} \frac{\partial}{\partial x_1} + \dots + a_{in} \frac{\partial}{\partial x_n},$$

$i = 1, \dots, n$. If $w_1, \dots, w_n \in R = \mathbb{K}[x_1, \dots, x_n]$ then by $J(w_1, \dots, w_n)$ will be denoted the Jacoby matrix of these polynomials.

2. ON NILPOTENT BASIC SUBALGEBRAS

Proposition 1. *Let L be a basic Lie subalgebra in $W_n(\mathbb{K})$ and d an element of L . Then the trace of d in L by the adjoint representation is equal to its divergence taken with the opposite sign, i.e. $\text{tr } d = -\text{div } d$.*

Proof. Let $\{D_1, D_2, \dots, D_n\}$ be an arbitrary basis of the Lie algebra L over \mathbb{K} . Let $D_i = a_{i1} \frac{\partial}{\partial x_1} + \dots + a_{in} \frac{\partial}{\partial x_n}$ be a decomposition of D_i in standard basis, $a_{ij} \in R = \mathbb{K}[x_1, x_2, \dots, x_n]$. Since $\{D_1, D_2, \dots, D_n\}$ is a basis of the free R -module $W_n(\mathbb{K})$, it holds obviously $\Delta = \det(a_{ij}) \in \mathbb{K}^*$. By Lemma 2.3 from [1], the divergences $\text{div } D_i = \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}$ and traces $\text{tr}(\text{ad } D_i) = \sum_{j=1}^n c_{ij}^j$ satisfy the following relation: $\text{tr}(\text{ad } D_i) = D_i(\Delta) - \text{div } D_i$. Since $D_i(\Delta) = 0$ we have $\text{tr}(\text{ad } D_i) = -\text{div } D_i$. The element d is a linear combination of D_i with coefficients in \mathbb{K} therefore $\text{tr } d = -\text{div } d$. \square

Corollary 1. *If L is a semisimple or nilpotent basic subalgebra of $W_n(\mathbb{K})$ then $L \in SW_n(\mathbb{K})$, where $SW_n(\mathbb{K})$ is the Lie algebra of all $D \in W_n(\mathbb{K})$ such that $\text{div } D = 0$.*

Lemma 1. *Let $\{D_1, D_2, \dots, D_n\}$ be an arbitrary basis of the R -module $W_n(\mathbb{K})$ and $[D_i, D_j] = \sum_{k=1}^n c_{ij}^k D_k$ for some $c_{ij}^k \in R = \mathbb{K}[x_1, x_2, \dots, x_n]$. Write down $\frac{\partial}{\partial x_i} = \sum_{j=1}^n b_{ij} D_j$ for all $i = 1, \dots, n$, where $b_{ij} \in R$. Then*

$$(1) \quad \sum_{i,j=1}^n b_{pi} b_{qj} c_{ij}^k + \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} = 0$$

for any $p, q, k = 1, \dots, n$.

Conversely, let $\{D_1, D_2, \dots, D_n\}$ be a basis of the R -module $W_n(\mathbb{K})$. Define the elements b_{ij} from the equations $\frac{\partial}{\partial x_i} = \sum_{j=1}^n b_{ij} D_j$. If these elements satisfy relations (1) for some $c_{ij}^k \in R$, then $[D_i, D_j] = \sum_{k=1}^n c_{ij}^k D_k$.

Proof. Using commutativity of the standard basis, we get:

$$(2) \quad 0 = \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} - \frac{\partial}{\partial x_q} \frac{\partial}{\partial x_p} = \frac{\partial}{\partial x_p} \sum_{j=1}^n b_{qj} D_j - \frac{\partial}{\partial x_q} \sum_{j=1}^n b_{pj} D_j$$

for any $p, q = 1, \dots, n$.

Observe that $\frac{\partial}{\partial x_p} \sum_{j=1}^n b_{qj} D_j = \sum_{j=1}^n b_{qj} \frac{\partial}{\partial x_p} D_j + \sum_{j=1}^n \frac{\partial b_{qj}}{\partial x_p} D_j$, analogously, $\frac{\partial}{\partial x_q} \sum_{j=1}^n b_{pj} D_j = \sum_{j=1}^n b_{pj} \frac{\partial}{\partial x_q} D_j + \sum_{j=1}^n \frac{\partial b_{pj}}{\partial x_q} D_j$. If we combine

this with (2), we have the relation

$$0 = \sum_{j=1}^n b_{qj} \frac{\partial}{\partial x_p} D_j - \sum_{i=1}^n b_{pi} \frac{\partial}{\partial x_q} D_i + \sum_{k=1}^n \left(\frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} \right) D_k.$$

Substituting $\sum_{j=1}^n b_{pj} D_j$ and $\sum_{j=1}^n b_{qj} D_j$ instead of $\frac{\partial}{\partial x_p}$ and respectively $\frac{\partial}{\partial x_q}$, we get the equality:

$$0 = \sum_{j=1}^n b_{qj} \left(\sum_{i=1}^n b_{pi} D_i \right) D_j - \sum_{i=1}^n b_{pi} \left(\sum_{j=1}^n b_{qj} D_j \right) D_i + \sum_{k=1}^n \left(\frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} \right) D_k.$$

It is easy to see that

$$\sum_{j=1}^n b_{qj} \left(\sum_{i=1}^n b_{pi} D_i \right) D_j - \sum_{i=1}^n b_{pi} \left(\sum_{j=1}^n b_{qj} D_j \right) D_i = \sum_{i,j=1}^n b_{pi} b_{qj} [D_i, D_j].$$

Combining this with the decomposition $[D_i, D_j] = \sum_{k=1}^n c_{ij}^k D_k$, we obtain the relation:

$$\sum_{i,j,k=1}^n b_{pi} b_{qj} c_{ij}^k D_k + \sum_{k=1}^n \left(\frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} \right) D_k = 0.$$

The derivations $\{D_1, D_2, \dots, D_n\}$ are linearly independent over R , therefore we have relation (1). This completes the proof of the first part of our statement.

To prove the second part of the Lemma we define elements b_{ij} by the next relations:

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^n b_{ij} D_j, \quad i, j = 1, \dots, n.$$

Suppose the elements c_{ij}^k satisfy relations (1). We have $[D_i, D_j] = \sum_{k=1}^n \gamma_{ij}^k D_k$ for some $\gamma_{ij}^k \in R$, because $\{D_1, D_2, \dots, D_n\}$ is a basis of the R -module $W_n(\mathbb{K})$. By the first part of this Lemma the elements γ_{ij}^k satisfy the relations (1), that is, $\sum_{i,j=1}^n b_{pi} b_{qj} \gamma_{ij}^k + \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} = 0$. The system (1) can be regarded as a linear system in n^3 variables c_{ij}^k . This system can be decomposed into a direct sum of n linear systems:

$$(3) \quad \sum_{i,j=1}^n b_{pi} b_{qj} c_{ij}^k + \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} = 0, \quad k \text{ is fixed, } k = 1 \dots n.$$

Let us prove that the system (3) has a unique solution. It is easy to see, that this system has the following matrix

$$\begin{pmatrix} b_{11}b_{11} & b_{11}b_{12} & \dots & b_{11}b_{1n} & \dots & b_{1n}b_{11} & b_{1n}b_{12} & \dots & b_{1n}b_{1n} \\ b_{11}b_{21} & b_{11}b_{22} & \dots & b_{11}b_{2n} & \dots & b_{1n}b_{21} & b_{1n}b_{22} & \dots & b_{1n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ b_{11}b_{n1} & b_{11}b_{n2} & \dots & b_{11}b_{nn} & \dots & b_{1n}b_{n1} & b_{1n}b_{n2} & \dots & b_{1n}b_{nn} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ b_{n1}b_{n1} & b_{n1}b_{n2} & \dots & b_{n1}b_{nn} & \dots & b_{nn}b_{n1} & b_{nn}b_{n2} & \dots & b_{nn}b_{nn} \end{pmatrix}.$$

Because this matrix is obviously the tensor square $(b_{ij}) \otimes (b_{ij})$ of the matrix (b_{ij}) and the determinant $\det(b_{ij})$ is invertible (because (b_{ij}) is a transition matrix between two bases), it holds $\det((b_{ij}) \otimes (b_{ij})) = (\det(b_{ij}))^{2n} \in \mathbb{K}^*$. Therefore, the system (3) has the unique solution $\gamma_{ij}^k = c_{ij}^k$ \square

Now let L be an arbitrary n -dimensional Lie algebra over the field \mathbb{K} . Take a basis $\{l_1, \dots, l_n\}$ of algebra L and denote by c_{ij}^k the structure constants of L in this basis, that is $[l_i, l_j] = \sum_{k=1}^n c_{ij}^k l_k$. It is well known that the tensor product $R \otimes_{\mathbb{K}} L$ of an associative and commutative \mathbb{K} -algebra R and the Lie algebra L is a Lie algebra over the field \mathbb{K} . Further, we will always denote by R the polynomial algebra $\mathbb{K}[x_1, \dots, x_n]$. Since the elements of the algebra $R \otimes_{\mathbb{K}} L$ are of the form $\sum_{i=1}^n (f_i \otimes l_i)$, $f_i \in R$, $i = 1, \dots, n$, the tensor product $R \otimes_{\mathbb{K}} L$ is a free module of rank n over the ring R . The elements $\{1 \otimes l_1, \dots, 1 \otimes l_n\}$ form obviously a basis of this module. Using the multiplication law in L , we get the equality

$$(4) \quad [\bar{f}, \bar{g}] = \sum_{k=1}^n \left(\sum_{i,j=1}^n c_{ij}^k f_i g_j \right) \otimes l_k.$$

for any elements $\bar{f} = \sum_{i=1}^n f_i \otimes l_i$, $\bar{g} = \sum_{i=1}^n g_i \otimes l_i \in R \otimes_{\mathbb{K}} L$.

For an arbitrary element $\bar{f} = \sum_{i=1}^n f_i \otimes l_i \in R \otimes_{\mathbb{K}} L$ and an arbitrary index $p = 1, \dots, n$ define $\frac{\partial \bar{f}}{\partial x_p} = \sum_{i=1}^n \frac{\partial f_i}{\partial x_p} \otimes l_i$. It is easy to see that the map $\bar{f} \mapsto \frac{\partial \bar{f}}{\partial x_p}$ is a derivation of the Lie algebra $R \otimes_{\mathbb{K}} L$ (we will denote this map also by $\frac{\partial}{\partial x_p}$). Since the derivation $\frac{\partial}{\partial x_p}$ acts on the coordinates f_i of the element $\bar{f} = \sum_{i=1}^n f_i \otimes l_i$, it holds $\frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} - \frac{\partial}{\partial x_q} \frac{\partial}{\partial x_p} = 0$ for arbitrary $p, q = 1, \dots, n$. Denote by A the abelian Lie subalgebra of $\text{Der}(R \otimes_{\mathbb{K}} L)$ with the basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$ and by \hat{L} the subalgebra $\hat{L} = A + R \otimes_{\mathbb{K}} L$ of the semidirect product of Lie algebras $\text{Der}(R \otimes_{\mathbb{K}} L) \ltimes R \otimes_{\mathbb{K}} L$.

Remark 1. Let L be a basic subalgebra of the Lie algebra $W_n(\mathbb{K})$. Then the equations (1) are equivalent to the following relations in the Lie algebra \hat{L} :

$$(5) \quad [\bar{b}_p, \bar{b}_q] + \left[\frac{\partial}{\partial x_p}, \bar{b}_q \right] - \left[\frac{\partial}{\partial x_q}, \bar{b}_p \right] = 0.$$

Since $\left[\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right] = 0$, we can rewrite relations (5) as the following relations in the Lie algebra \hat{L}

$$(6) \quad \left[\frac{\partial}{\partial x_p} + \bar{b}_p, \frac{\partial}{\partial x_q} + \bar{b}_q \right] = 0.$$

Let L be a nilpotent Lie algebra over the field \mathbb{K} with $\dim L_{\mathbb{K}} = n$. By Engel's theorem, L has a flag of ideals $L = L_0 \supseteq L_1 \supseteq \dots \supseteq L_{n-1} \supseteq L_n = \{0\}$. Take any elements $l_i \in L_{i-1} \setminus L_i$, $i = 1, \dots, n$ and consider the Lie algebra \hat{L} constructed in such a way as it was mentioned above. The structure constants of L in the basis $\{l_1, \dots, l_n\}$ satisfy the relations

$$(7) \quad c_{ij}^k = 0, \text{ if } k \leq \max(i, j).$$

Since the Lie algebra L is nilpotent, the tensor product $R \otimes_{\mathbb{K}} L$ is also nilpotent. Then for any element $\bar{w} \in R \otimes_{\mathbb{K}} L$ the inner derivation $\text{ad } \bar{w}$ of the Lie algebra \hat{L} is nilpotent. We collect some properties of the Lie algebras $R \otimes_{\mathbb{K}} L$ and \hat{L} in the following Lemma.

Lemma 2. *Let L be a nilpotent Lie algebra of dimension n over the field \mathbb{K} . Let $\{l_1, \dots, l_n\}$ be a basis of L and $\bar{w} = \sum_{i=1}^n w_i \otimes l_i$ be an arbitrary element of $R \otimes_{\mathbb{K}} L$. Then it holds*

- (1) $\text{ad } \bar{w}$ is a nilpotent endomorphism of the R -module $R \otimes_{\mathbb{K}} L$, that is $\text{ad } \bar{w}(f\bar{u}) = f \text{ad } \bar{w}(\bar{u})$ for any $f \in R$ and $\bar{u} = \sum_{i=1}^n u_i \otimes l_i \in R \otimes_{\mathbb{K}} L$;
- (2) $\varphi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \bar{w})^{i-1}$ is an automorphism of the R -module $R \otimes_{\mathbb{K}} L$;
- (3) if $\bar{w} = \sum_{i=1}^n w_i \otimes l_i$ is an element of $R \otimes_{\mathbb{K}} L$ with the property $\det J(w_1, \dots, w_n) = c \in \mathbb{K}^*$, then the set of elements

$$\bar{b}_p = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left(\frac{\partial w}{\partial x_p} \right), \quad p = 1, \dots, n$$

is a basis of the free R -module $R \otimes_{\mathbb{K}} L$. In particular, writing $\bar{b}_p = \sum_{i=1}^n b_{pi} \otimes l_i$ we have $\det(b_{ij})_{i,j=1}^n = c \in \mathbb{K}^*$.

Proof. (1) Obvious.

(2) As $\text{ad } \bar{w}$ is a nilpotent endomorphism of R -module $R \otimes_{\mathbb{K}} L$, the map $\varphi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \bar{w})^{i-1}$ is well defined and is an endomorphism of the R -module $R \otimes_{\mathbb{K}} L$. Since $\varphi = E + \sum_{i=2}^{\infty} \frac{1}{i!} (\text{ad } \bar{w})^{i-1}$ is the sum of the identity and a nilpotent endomorphisms, the map φ is an automorphism of the free R -module $R \otimes_{\mathbb{K}} L$.

(3) Let $\bar{w} = \sum_{i=1}^n w_i \otimes l_i$ be an element of $R \otimes_{\mathbb{K}} L$ such that $\det J(w_1, \dots, w_n) = c \in \mathbb{K}^*$. It is easy to see that the set of elements $\{\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n}\}$ is a basis of the R -module $R \otimes_{\mathbb{K}} L$. Since the map φ defined above is an automorphism of the R -module $R \otimes_{\mathbb{K}} L$, the set of elements $\bar{b}_p = \varphi(\frac{\partial w}{\partial x_p})$, $p = 1, \dots, n$ is the basis of this module. Therefore $\det(b_{ij})_{i,j=1}^n = c \in \mathbb{K}^*$. \square

Theorem 1. (1) *Let L be an arbitrary nilpotent Lie algebra over any field \mathbb{K} of characteristic 0. Then there exists a basic subalgebra \bar{L} of $W_n(\mathbb{K})$, such that \bar{L} is isomorphic to L (by that every basis of \bar{L} over \mathbb{K} is a basis of R -module $W_n(\mathbb{K})$).*

(2) *Let \bar{L} be a basic Lie subalgebra of $W_n(\mathbb{K})$, such that \bar{L} is isomorphic to a nilpotent Lie algebra L with a basis $\{l_1, l_2, \dots, l_n\}$, satisfying the relations (7), let D_i be the images of the elements l_i by this isomorphism. Write down $\frac{\partial}{\partial x_i} = \sum_{j=1}^n b_{ij} D_j$, $\bar{b}_p = \sum_{i=1}^n b_{pi} \otimes l_i$, $p = 1, \dots, n$. Then there exists an element $\bar{w} = \sum_{i=1}^n w_i \otimes l_i \in R \otimes_{\mathbb{K}} L$ such that $\det J(w_1, w_2, \dots, w_n) \in \mathbb{K}^*$ and the following equalities hold:*

$$\bar{b}_p = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left(\frac{\partial w}{\partial x_p} \right), \quad p = 1, \dots, n$$

(the number of nonzero summands in this series is finite because the Lie algebra $R \otimes_{\mathbb{K}} L$ is nilpotent).

Proof. (1) Let L be a nilpotent Lie algebra over the field \mathbb{K} of dimension n and $\{l_1, l_2, \dots, l_n\}$ be its basis such that the structure constants c_{ij}^k satisfy the

relations (7). We will construct $b_{ij} \in R, i, j = 1, \dots, n$, which satisfy relations (1) and have property $\det(b_{ij})_{i,j=1}^n \in \mathbb{K}^*$. If we consider the Lie algebra \hat{L} , then by Remark 1 the conditions (1) are equivalent to the conditions (6) in the terms of \hat{L} for $\bar{b}_p = \sum_{i=1}^n b_{pi} \otimes l_i, p = 1, \dots, n$:

$$\left[\frac{\partial}{\partial x_p} + \bar{b}_p, \frac{\partial}{\partial x_q} + \bar{b}_q \right] = 0.$$

Take an arbitrary element $\bar{w} = \sum_{i=1}^n w_i \otimes l_i \in \hat{L}$. Put $\bar{b}_p = \sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left(-\frac{\partial \bar{w}}{\partial x_p} \right), p = 1, \dots, n$, where as usually $(\text{ad } \bar{w})^0 = E$ is the identity operator. Note, that the relations

$$\sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left(-\frac{\partial \bar{w}}{\partial x_p} \right) = \sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left[\bar{w}, \frac{\partial}{\partial x_p} \right] = \sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^i \left(\frac{\partial}{\partial x_p} \right)$$

hold in the Lie algebra \hat{L} for $p = 1, \dots, n$. Therefore, we have the following equalities in the Lie algebra \hat{L} :

$$\begin{aligned} & \left[\frac{\partial}{\partial x_p} + \bar{b}_p, \frac{\partial}{\partial x_q} + \bar{b}_q \right] = \\ & \left[\frac{\partial}{\partial x_p} + \sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^i \left(\frac{\partial}{\partial x_p} \right), \frac{\partial}{\partial x_q} + \sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^i \left(\frac{\partial}{\partial x_q} \right) \right] = \\ & \left[e^{\text{ad } \bar{w}} \left(\frac{\partial}{\partial x_p} \right), e^{\text{ad } \bar{w}} \left(\frac{\partial}{\partial x_q} \right) \right] = e^{\text{ad } \bar{w}} \left[\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right] = 0 \end{aligned}$$

(we took into account that $e^{\text{ad } \bar{w}}$ is an automorphism of the Lie algebra \hat{L} and $\left[\frac{\partial}{\partial x_q}, \frac{\partial}{\partial x_p} \right] = 0, p, q = 1, \dots, n$).

Thus, we have elements $\bar{b}_1 = \sum_{i=1}^n b_{1i} \otimes l_i, \bar{b}_2 = \sum_{i=1}^n b_{2i} \otimes l_i, \dots, \bar{b}_n = \sum_{i=1}^n b_{ni} \otimes l_i$ satisfying equations (5), then these elements satisfy also the equations (1).

Take any element \bar{w} of $R \otimes L$ such that $\det J(\bar{w}) \in \mathbb{K}^*$, for example, $\bar{w} = \sum_{i=1}^n x_i \otimes l_i \in \hat{L}$. Let $B^{-1} = A = (a_{ij})$ be the inverse matrix for B (the matrix $B = (b_{ij})$ is invertible by Lemma 2 p.3). By Lemma 1, the derivations $D_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}, i = 1, \dots, n$, satisfy the relations $[D_i, D_j] = \sum_{k=1}^n c_{ij}^k D_k$ and $\det(a_{ij}) \in \mathbb{K}^*$. We have shown that $\mathbb{K}\langle D_1, D_2, \dots, D_n \rangle$ is a basic subalgebra of $W_n(\mathbb{K})$, which is isomorphic to L . This completes the proof of part (1).

(2) Take any elements

$$\bar{b}_1 = \sum_{i=1}^n b_{1i} \otimes l_i, \bar{b}_2 = \sum_{i=1}^n b_{2i} \otimes l_i, \dots, \bar{b}_n = \sum_{i=1}^n b_{ni} \otimes l_i$$

of $R \otimes_{\mathbb{K}} L$ satisfying the relations (6). We will build an element $\bar{w} \in R \otimes_{\mathbb{K}} L$ such that $\bar{b}_p = \sum_{i=1}^\infty \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left(-\frac{\partial \bar{w}}{\partial x_p} \right), p = 1, \dots, n$. This equality is equivalent to the relation $e^{\text{ad } \bar{w}} \left(\frac{\partial}{\partial x_p} \right) = \frac{\partial}{\partial x_p} + \bar{b}_p$. Applying the automorphism $e^{\text{ad } -\bar{w}}$ to the both sides of this relation, we obtain

$$(8) \quad \frac{\partial}{\partial x_p} = e^{\text{ad } -\bar{w}} \left(\frac{\partial}{\partial x_p} + \bar{b}_p \right).$$

Thus, we must prove that this equality holds for $p = 1, \dots, n$. Consider the matrix $B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$ with entries that are coordinates of the elements $\bar{b}_1, \dots, \bar{b}_n$. Let m_1 be the number of the first nonzero column of B (for $B = 0$ take $w = 0$, then (8) obviously holds). Write the equations (1) for $k = m_1$:

$$(9) \quad \sum_{i,j=1}^n b_{pi} b_{qj} c_{ij}^{m_1} + \frac{\partial b_{qm_1}}{\partial x_p} - \frac{\partial b_{pm_1}}{\partial x_q} = 0, \quad p, q = 1, \dots, n.$$

Since m_1 is the number of the first nonzero column of B , we have $b_{pi} = 0$ for $i < m_1$, $p = 1, \dots, n$. On the other hand, if $i \geq m_1$, then $c_{ij}^k = 0$ by (7) (because L is nilpotent). Hence (9) is equivalent to the equations $\frac{\partial b_{qm_1}}{\partial x_p} - \frac{\partial b_{pm_1}}{\partial x_q} = 0$, $p, q = 1, \dots, n$. Then there exists a polynomial h_1 such that $b_{pm_1} = \frac{\partial h}{\partial x_p}$, $p = 1, \dots, n$ (see, for example [6]). Denote $\bar{h}_1 = h_1 \otimes l_{m_1}$ and consider the elements $e^{\text{ad } \bar{h}_1} \left(\frac{\partial}{\partial x_p} + \bar{b}_p \right)$ $p = 1, \dots, n$. Note that

$$\bar{b}_p = \sum_{i=m_1}^n b_{pi} \otimes l_i = \frac{\partial h}{\partial x_p} \otimes l_{m_1} + \sum_{i=m_1+1}^n b_{pi} \otimes l_i, \quad p = 1, \dots, n,$$

and therefore it holds

$$e^{\text{ad } \bar{h}_1} \left(\frac{\partial}{\partial x_p} + \bar{b}_p \right) = e^{\text{ad } \bar{h}_1} \left(\frac{\partial}{\partial x_p} \right) + e^{\text{ad } \bar{h}_1} (\bar{b}_p) =$$

$$\frac{\partial}{\partial x_p} - \frac{\partial h_1}{\partial x_p} \otimes l_{m_1} + e^{\text{ad } \bar{h}_1} \left(\frac{\partial h}{\partial x_p} \otimes l_{m_1} + \sum_{i=m_1+1}^n b_{pi} \otimes l_i \right), \quad p = 1, \dots, n.$$

Since $\left[\bar{h}_1, \frac{\partial h}{\partial x_p} \otimes l_{m_1} \right] = 0$, we get $e^{\text{ad } \bar{h}_1} \left(\frac{\partial h}{\partial x_p} \otimes l_{m_1} \right) = \frac{\partial h_1}{\partial x_p} \otimes l_{m_1}$, $p = 1, \dots, n$.

It is easy to see that $e^{\text{ad } \bar{h}_1} \left(\sum_{i=m_1+1}^n b_{pi} \otimes l_i \right) \in R \otimes \langle l_{m_1+1}, \dots, l_n \rangle$, because $R \otimes \langle l_{m_1+1}, \dots, l_n \rangle$ is an ideal of the algebra $R \otimes L$. Then we have

$$e^{\text{ad } \bar{h}_1} \left(\frac{\partial}{\partial x_p} + \bar{b}_p \right) = \frac{\partial}{\partial x_p} - \frac{\partial h_1}{\partial x_p} \otimes l_{m_1} + \frac{\partial h_1}{\partial x_p} \otimes l_{m_1} + \bar{d}_p,$$

for $\bar{d}_p = e^{\text{ad } \bar{h}_1} \left(\sum_{i=m_1+1}^n b_{pi} \otimes l_i \right) \in R \otimes \langle l_{m_1+1}, \dots, l_n \rangle$. Therefore,

$$e^{\text{ad } \bar{h}_1} \left(\frac{\partial}{\partial x_p} + \bar{b}_p \right) = \frac{\partial}{\partial x_p} + \bar{d}_p, \quad p = 1, \dots, n.$$

Denote by d_{pi} the coordinates of the element \bar{d}_p in basis $\{1 \otimes l_1, \dots, 1 \otimes l_n\}$, $p = 1, \dots, n$, i.e. $\bar{d}_p = \sum_{i=1}^n d_{pi} \otimes l_i$. Consider the matrix $D = \begin{pmatrix} d_{11} & \dots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \dots & d_{nn} \end{pmatrix}$. We have just proved that the first nonzero column of the matrix D has the number $m_2 > m_1$. Analogously, applying the

automorphism $e^{\text{ad } \bar{h}_1}$ to the elements $\frac{\partial}{\partial x_p} + \bar{d}_p$, $p = 1, \dots, n$, we get the elements $\frac{\partial}{\partial x_p} + \bar{f}_p$. Define the elements f_{ij} , $i, j = 1, \dots, n$ from the equalities $\bar{f}_p = \sum_{i=1}^n f_{pi} \otimes l_i$. The first nonzero column in the matrix $F = \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \dots & f_{nn} \end{pmatrix}$ has the number $m_3 > m_2$. It is obvious that after s steps ($s \leq n$) we get the elements $\frac{\partial}{\partial x_p}$, $p = 1, \dots, n$ with the corresponding zero matrix. Therefore $e^{\text{ad } \bar{h}_s} \dots e^{\text{ad } \bar{h}_1} \left(\frac{\partial}{\partial x_p} + \bar{b}_p \right) = \frac{\partial}{\partial x_p}$, $p = 1, \dots, n$. Since the Lie algebra $R \otimes L$ is nilpotent, there exists (by Campbell-Baker-Hausdorff formula) an element $w \in R \otimes L$ such that $e^{\text{ad } \bar{h}_s} \dots e^{\text{ad } \bar{h}_1} = e^{\text{ad } -\bar{w}}$. Finally, let \bar{L} be a basic subalgebra which is isomorphic to L , $\{D_1, \dots, D_n\}$ be a basis of \bar{L} . Write $\frac{\partial}{\partial x_p} = \sum_{j=1}^n b_{pj} D_j$, the isomorphism is defined by the map $l_i \mapsto D_i$. Then $\bar{b}_p = \sum_{i=1}^n b_{pi} \otimes l_i$, satisfies (1), therefore $\bar{b}_p = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \bar{w})^{i-1} \left(\frac{\partial \bar{w}}{\partial x_p} \right)$ for an element $\bar{w} \in R \otimes_{\mathbb{K}} L$ such that $\det J(\bar{w})$ is invertible. This completes the proof of the theorem. \square

Example 1. Let $L = H_n$ be the $2n+1$ -dimensional Heisenberg Lie algebra, $\{l_1, \dots, l_{2n+1}\}$ be its standard basis with multiplication rule $[l_i, l_{n+i}] = l_{2n+1}$ for $1 \leq i \leq n$, (other products are zero). Then, in this basis $c_{i,n+i}^{2n+1} = 1$, $c_{n+i,i}^{2n+1} = -1$, $1 \leq i \leq n$, other structure constants are zero. Take $\bar{w} = \sum_{i=1}^n -x_i \otimes l_i$. It is clear that $\frac{\partial \bar{w}}{\partial x_p} = -1 \otimes l_p$ and

$$(10) \quad \bar{b}_p = 1 \otimes l_p - \frac{1}{2} \left[\sum_{i=1}^n x_i \otimes l_i, 1 \otimes l_p \right], 1 \leq p \leq n$$

in the Lie algebra \hat{L} . Easy calculation shows that $\bar{b}_p = 1 \otimes l_p + \frac{1}{2} x_{p+n} \otimes l_{2n+1}$, $p \leq n$, $\bar{b}_p = 1 \otimes l_p - \frac{1}{2} x_{p-n} \otimes l_{2n+1}$, $n < p \leq 2n$ and $\bar{b}_{2n+1} = 1 \otimes l_{2n+1}$. It can be easily shown that $\det(b_{ij}) = 1$. Passing to the inverse matrix B^{-1} to the matrix $B = (b_{pi})_{p,i=1}^n$ one can easily show that the linear span over \mathbb{K} of the following derivations is a basic subalgebra of $W_{2n+1}(\mathbb{K})$ which is isomorphic to H_n :

$$D_i = \frac{\partial}{\partial x_i} - \frac{1}{2} x_{n+i} \frac{\partial}{\partial x_{2n+1}}, 1 \leq i \leq n,$$

$$D_i = \frac{\partial}{\partial x_i} + \frac{1}{2} x_{n-i} \frac{\partial}{\partial x_{2n+1}}, n < i \leq 2n, \quad D_{2n+1} = \frac{\partial}{\partial x_{2n+1}}$$

3. ON THE SOLVABLE BASIC LIE SUBALGEBRAS

Some known properties of finite dimensional Lie algebras and modules over them are collected in the next Lemma.

Lemma 3. *Let L be a finite dimensional Lie algebra over an algebraically closed field of zero characteristic and let H be any its Cartan subalgebra. Then*

- (1) *if the algebra L is solvable, then $L = H + [L, L]$;*

(2) if L is semisimple and $L = N_- \oplus H \oplus N_+$ is its triangular decomposition, then the subalgebras N_+ and N_- act nilpotently on every finite dimensional module M over the Lie algebra L .

Proposition 2. *Let L be an arbitrary n -dimensional ($n \geq 1$) solvable Lie algebra over an algebraically closed field \mathbb{K} of zero characteristic, then there is a basic subalgebra \overline{L} of $W_n(\mathbb{K})$, such that \overline{L} is isomorphic to L .*

Proof. Let H be any Cartan subalgebra of L . Take a basis $\{l_1, \dots, l_n\}$ of L with the following property: $\{l_1, \dots, l_m\}$ is a basis of H , and if $H \cap [L, L] \neq 0$, then $\{l_{k+1}, \dots, l_m\}$ is a basis of $H \cap [L, L]$, $m \geq k$, $\{l_{k+1}, \dots, l_n\}$ is a basis of $[L, L]$. Note that $[L, L]$ is a nilpotent ideal of L because L is solvable and the field \mathbb{K} has zero characteristic; the subalgebra H is nilpotent as a Cartan subalgebra of L .

Now, put $\overline{w} = \sum_{i=1}^k x_i \otimes l_i$. Consider the linear map $\phi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \overline{w})^{i-1}$ from the R -module $R \otimes_{\mathbb{K}} H$ to itself (since H is nilpotent, the sum is finite). By Lemma 2, ϕ is an automorphism of the R -module $R \otimes_{\mathbb{K}} H$. As $R \otimes_{\mathbb{K}} (H \cap [L, L])$ is an ideal of the algebra $R \otimes_{\mathbb{K}} H$, the map $\text{ad } \overline{w}$ saves the submodule $R \otimes_{\mathbb{K}} (H \cap [L, L])$. The set of elements $\{-1 \otimes l_i\}, i = 1, \dots, m$ is a basis of the R -module $R \otimes_{\mathbb{K}} H$, so it is obvious that $\{\phi(-1 \otimes l_i)\}, i = 1, \dots, m$ is also a basis of $R \otimes_{\mathbb{K}} H$. Further, the set $\{\phi(-1 \otimes l_i)\}, i = k+1, \dots, m$ is a basis of the submodule $R \otimes_{\mathbb{K}} (H \cap [L, L])$. Then, it is clear that the elements $\phi(-1 \otimes l_i), i = 1, \dots, k$ and $-1 \otimes l_i, i = k+1, \dots, m$ together form a basis of the R -module $R \otimes_{\mathbb{K}} H$.

Put $d_p = \phi(-1 \otimes l_i), i = 1, \dots, k$ and consider the automorphism $\theta = \exp \text{ad } w$ of the algebra \widehat{H} . Then we get for $p = 1, \dots, k$:

$$\begin{aligned} \theta \left(\frac{\partial}{\partial x_p} \right) &= \frac{\partial}{\partial x_p} + \left[w, \frac{\partial}{\partial x_p} \right] + \frac{1}{2!} \left[w, \left[w, \frac{\partial}{\partial x_p} \right] \right] + \dots = \\ &= \frac{\partial}{\partial x_p} - \frac{\partial w}{\partial x_p} - \frac{1}{2!} \left[w, \frac{\partial w}{\partial x_p} + \dots \right] = \frac{\partial}{\partial x_p} + \phi(-1 \otimes l_p) = \frac{\partial}{\partial x_p} + \overline{d}_p. \end{aligned}$$

It follows from this that

$\left[\frac{\partial}{\partial x_p} + \overline{d}_p, \frac{\partial}{\partial x_q} + \overline{d}_q \right] = \left[\theta \left(\frac{\partial}{\partial x_p} \right), \theta \left(\frac{\partial}{\partial x_q} \right) \right] = \theta \left(\left[\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right] \right) = 0, p, q = 1, \dots, k$. Consider the elements $\overline{d}_p, p = 1, \dots, k$ as elements of the algebra $R \otimes L$. Put $\overline{d}_p = 0, p = k+1, \dots, n$. It is clear, taking into account the choice of \overline{w} , that $\frac{\partial \overline{d}_p}{\partial x_q} = 0, q = k+1, \dots, n$. Therefore, $\left[\frac{\partial}{\partial x_p} + \overline{d}_p, \frac{\partial}{\partial x_q} + \overline{d}_q \right] = 0, p, q = 1, \dots, n$.

Now put $\overline{u} = \sum_{i=k+1}^n x_i \otimes l_i$. Consider the linear map ψ from the R -module $R \otimes L$ to $R \otimes L$: $\psi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \overline{u})^{i-1}$. (The sum is finite, because $R \otimes [L, L]$ acts nilpotently on $R \otimes L$ and $u \in R \otimes [L, L]$). By Lemma 2, ψ is an automorphism of the R -module $R \otimes L$. It is obvious that the elements $\overline{d}_p, p = 1, \dots, k$ and $-1 \otimes l_i, i = k+1, \dots, n$ form together a basis of the R -module $R \otimes L$. Note that the set $\{-1 \otimes l_i\}, i = k+1, \dots, n$ is a basis of the R -module $R \otimes [L, L]$. Therefore the elements $\psi(-1 \otimes l_i), i = k+1, \dots, n$ form a basis of the R -module $R \otimes [L, L]$.

Consider now the automorphism $\eta = e^{\text{ad } \overline{u}} = \sum_{i=0}^{\infty} \frac{1}{i!} (\text{ad } \overline{u})^i$ of the algebra \widehat{L} . Then, the elements $\eta(\overline{d}_p), p = 1, \dots, k$ and $\eta(-1 \otimes l_p), p = k+1, \dots, n$ form a basis of the R -module $R \otimes L$, and the set $\eta(-1 \otimes l_p), p = k+1, \dots, n$ is a basis

of the R -submodule $R \otimes [L, L]$. Note, that $\psi(-1 \otimes l_p)$, $p = k+1, \dots, n$ is also a basis of the R -submodule $R \otimes [L, L]$. Then, the elements $\eta(\overline{d_p})$, $p = 1, \dots, k$ and $\psi(-1 \otimes l_p)$, $p = k+1, \dots, n$ form together a basis of the R -module $R \otimes L$. Put $\overline{b_p} = \eta(\overline{d_p})$, $p = 1, \dots, k$ and $\overline{b_p} = \psi(-1 \otimes l_p)$, $p = k+1, \dots, n$. Writing down $\overline{b_p} = \sum_{i=1}^n (b_{pi} \otimes l_i)$, we have $\det(b_{pi})_{p,i=1}^n \in \mathbb{K}^*$.

By construction of the element \overline{u} , it holds $\left[\overline{u}, \frac{\partial}{\partial x_p}\right] = 0$ for $p = 1, \dots, k$, so we have $e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_p}\right) = \frac{\partial}{\partial x_p}$, $p = 1, \dots, k$. Note that

$$\frac{\partial}{\partial x_p} + \overline{b_p} = \frac{\partial}{\partial x_p} + \eta(\overline{d_p}) = e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_p}\right) + e^{\text{ad } \overline{u}}(\overline{d_p}) = e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_p} + \overline{d_p}\right) \quad p = 1, \dots, k.$$

It is easy to see, that for $p = k+1, \dots, n$ it holds

$$-1 \otimes l_p = -\frac{\partial \overline{u}}{\partial x_p} = \left[\overline{u}, \frac{\partial}{\partial x_p}\right].$$

Hence, the relations

$$\begin{aligned} \frac{\partial}{\partial x_p} + \overline{b_p} &= \frac{\partial}{\partial x_p} + \psi(-1 \otimes l_p) = \frac{\partial}{\partial x_p} + \left(E + \frac{1}{2!}(\text{ad } \overline{u}) + \frac{1}{3!}(\text{ad } \overline{u})^2 + \dots\right)(-1 \otimes l_p) = \\ &= \frac{\partial}{\partial x_p} + (-1 \otimes l_p) + \frac{1}{2!}[\overline{u}, -1 \otimes l_p] + \frac{1}{3!}[\overline{u}, [\overline{u}, -1 \otimes l_p]] + \dots = \\ &= \frac{\partial}{\partial x_p} + \left[\overline{u}, \frac{\partial}{\partial x_p}\right] + \frac{1}{2!} \left[\overline{u}, \left[\overline{u}, \frac{\partial}{\partial x_p}\right]\right] + \dots = e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_p}\right). \end{aligned}$$

hold for $p = k+1, \dots, n$. Since $\overline{d_p} = 0$ for $p = k+1, \dots, n$, we get

$$\frac{\partial}{\partial x_p} + \overline{b_p} = e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_p} + \overline{d_p}\right).$$

Thus,

$$\begin{aligned} \left[\frac{\partial}{\partial x_p} + \overline{b_p}, \frac{\partial}{\partial x_q} + \overline{b_q}\right] &= \left[e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_p} + \overline{d_p}\right), e^{\text{ad } \overline{u}} \left(\frac{\partial}{\partial x_q} + \overline{d_q}\right)\right] = \\ &= e^{\text{ad } \overline{u}} \left[\frac{\partial}{\partial x_p} + \overline{d_p}, \frac{\partial}{\partial x_q} + \overline{d_q}\right] = 0. \end{aligned}$$

Therefore, by Lemma 1 there exists a basic subalgebra of $W_n(K)$ which is isomorphic to L . □

4. THE MAIN THEOREM

Lemma 4. *Let L be n -dimensional Lie algebra over an algebraically closed field \mathbb{K} of zero characteristic. Let $L = L_1 + L_2$, where L_1, L_2 are subalgebras of L such that $L_1 \cap L_2 = \{0\}$, $\dim L_1 = m < n$. Assume that the subalgebra L_2 acts nilpotently (by means of multiplication) on L , that is $(\text{ad } L_2)^k(L) = 0$ for some k . If there exists a basic subalgebra \overline{L}_1 of $W_m(\mathbb{K})$ such that \overline{L}_1 is isomorphic to L_1 , then there exists a basic subalgebra of $W_n(\mathbb{K})$ such that \overline{L} is isomorphic to the Lie algebra L .*

Proof. Take a basis $\{l_1, \dots, l_m\}$ of the subalgebra L_1 and a basis $\{l_{m+1}, \dots, l_n\}$ of the subalgebra L_2 . Then the elements l_1, \dots, l_n form a basis of L . Denote by R_1 the subring $\mathbb{K}[x_1, \dots, x_m]$ of the polynomial ring $R = \mathbb{K}[x_1, \dots, x_m, x_{m+1}, \dots, x_n]$. Since there exists a basic subalgebra $\overline{L_1}$ of $W_m(\mathbb{K})$ such that $\overline{L_1}$ is isomorphic to L_1 , by Lemma (1) there exist elements $\overline{d_p} = \sum_{i=1}^m d_{pi} \otimes l_i \in R_1 \otimes_{\mathbb{K}} L_1$, $p = 1, \dots, m$ such that for the polynomials d_{pi} the relation (1) holds with the structure constants of the algebra L_1 and $\det(d_{pi})_{p,i=1}^m \in \mathbb{K}^*$. As L_1 is a subalgebra of L , we have a natural embedding of the Lie algebra $R_1 \otimes L_1$ into $R \otimes L$ (an element $\overline{d_p} = \sum_{i=1}^m d_{pi} \otimes l_i \in R_1 \otimes_{\mathbb{K}} L_1$ maps to the element $\overline{d_p} = \sum_{i=1}^n d_{pi} \otimes l_i \in R \otimes_{\mathbb{K}} L$, $d_{pi} = 0$ for $i = m+1, \dots, n$). Put $\overline{d_p} = 0$ for $p = m+1, \dots, n$. Using Remark 1, it is easy to see that the following relations hold in \widehat{L} :

$$(11) \quad \left[\frac{\partial}{\partial x_p} + \overline{d_p}, \frac{\partial}{\partial x_q} + \overline{d_q} \right] = 0, \quad p, q = 1, \dots, n.$$

Put $\overline{b} = \sum_{i=m+1}^n x_i \otimes l_i \in R \otimes L_2$. By the assumption for the subalgebra L_2 , the derivation $\text{ad } \overline{b}$ of the Lie algebra $R \otimes L$ is nilpotent and therefore the automorphism $\theta = \exp(\text{ad } \overline{b})$ of the Lie algebra $R \otimes L$ (and \widehat{L}) is well defined. Denote $\overline{b_p} = -\frac{\partial}{\partial x_p} + \theta \left(\frac{\partial}{\partial x_p} + \overline{d_p} \right)$, $p = 1 \dots n$. Then $\frac{\partial}{\partial x_p} + \overline{b_p} = \theta \left(\frac{\partial}{\partial x_p} + \overline{d_p} \right)$, and the following equalities hold:

$$\begin{aligned} \left[\frac{\partial}{\partial x_p} + \overline{b_p}, \frac{\partial}{\partial x_q} + \overline{b_q} \right] &= \left[\theta \left(\frac{\partial}{\partial x_p} + \overline{d_p} \right), \theta \left(\frac{\partial}{\partial x_q} + \overline{d_q} \right) \right] = \\ &= \theta \left(\left[\frac{\partial}{\partial x_p} + \overline{d_p}, \frac{\partial}{\partial x_q} + \overline{d_q} \right] \right) = 0, \quad p, q = 1 \dots n. \end{aligned}$$

Let us show that the set of the elements $\overline{b_p}$, $p = 1, \dots, n$ is a basis of the free R -module $R \otimes L$. It holds

$$\overline{b_p} = -\frac{\partial}{\partial x_p} + \theta \left(\frac{\partial}{\partial x_p} + \overline{d_p} \right) = -\frac{\partial}{\partial x_p} + \theta \left(\frac{\partial}{\partial x_p} \right) + \theta(\overline{d_p}) \quad p = 1, \dots, m$$

and then

$$\begin{aligned} \theta \left(\frac{\partial}{\partial x_p} \right) &= \left(E + \text{ad } \overline{b} + \frac{1}{2!}(\text{ad } \overline{b})^2 + \dots \right) \left(\frac{\partial}{\partial x_p} \right) = \\ &= \frac{\partial}{\partial x_p} + \left[\overline{b}, \frac{\partial}{\partial x_p} \right] + \left[\overline{b}, \left[\overline{b}, \frac{\partial}{\partial x_p} \right] \right] + \dots = \frac{\partial}{\partial x_p}, \end{aligned}$$

since $\left[\overline{b}, \frac{\partial}{\partial x_p} \right] = -\frac{\partial \overline{b}}{\partial x_p} = 0$ for $p = 1, \dots, m$. Now consider the elements $\overline{b_p}$ for $p = m+1, \dots, n$. In this case $\overline{d_p} = 0$. Therefore

$$\begin{aligned} \overline{b_p} &= -\frac{\partial}{\partial x_p} + \theta \left(\frac{\partial}{\partial x_p} + \overline{d_p} \right) = -\frac{\partial}{\partial x_p} + \left(E + \text{ad } \overline{b} + \frac{1}{2!}(\text{ad } \overline{b})^2 + \dots \right) \left(\frac{\partial}{\partial x_p} \right) = \\ &= -\frac{\partial}{\partial x_p} + \frac{\partial}{\partial x_p} + \left[\overline{b}, \frac{\partial}{\partial x_p} \right] + \left[\overline{b}, \left[\overline{b}, \frac{\partial}{\partial x_p} \right] \right] + \dots = \\ &= -\frac{\partial \overline{b}}{\partial x_p} - \frac{1}{2!} \left[\overline{b}, \frac{\partial \overline{b}}{\partial x_p} \right] - \frac{1}{3!} \left[\overline{b}, \left[\overline{b}, \frac{\partial \overline{b}}{\partial x_p} \right] \right] = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \overline{b})^{i-1} \left(-\frac{\partial \overline{b}}{\partial x_p} \right). \end{aligned}$$

(Because of nilpotency of $\text{ad } \bar{b}$, the number of nonzero summands in this series is finite).

Denote $\phi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad } \bar{b})^{i-1}$. It is easy to see that ϕ is an automorphism of the free R -module $R \otimes L$ (see Lemma 2). Since $\bar{b} = \sum_{i=m+1}^n x_i \otimes l_i \in R \otimes L_2$, the R -module $R \otimes L_2$ is invariant under action of ϕ .

The set of elements \bar{d}_p , $p = 1, \dots, m$ and $-\frac{\partial \bar{b}}{\partial x_p} = -1 \otimes l_p$, $p = m+1, \dots, n$ is a basis of the free R -module $R \otimes L$ (because this module is the direct sum of the R -modules $R \otimes L_1$ and $R \otimes L_2$). Then, the elements $\theta(\bar{d}_p)$, $p = 1, \dots, m$ and $\theta\left(-\frac{\partial \bar{b}}{\partial x_p}\right)$, $p = m+1, \dots, n$ form a basis of $R \otimes L$. Since ϕ is an automorphism of the free R -module $R \otimes L_2$, the set of elements $\phi\left(-\frac{\partial \bar{b}}{\partial x_p}\right)$, $p = m+1, \dots, n$ is a basis of this submodule (note that the set of elements $\theta\left(-\frac{\partial \bar{b}}{\partial x_p}\right)$, $p = m+1, \dots, n$ is also a basis of $R \otimes L_2$). It follows from this that the elements $\theta(\bar{d}_p)$, $p = 1, \dots, m$ and $\phi\left(-\frac{\partial \bar{b}}{\partial x_p}\right)$, $p = m+1, \dots, n$ together form a basis of the free R -module $R \otimes L$. Then, using the equalities $\bar{b}_p = \theta(\bar{d}_p)$, $p = 1, \dots, m$ and $\bar{b}_p = \phi\left(-\frac{\partial \bar{b}}{\partial x_p}\right)$, $p = m+1, \dots, n$, we see that $\{\bar{b}_p\}$, $p = 1, \dots, n$ is a basis of the free R -module $R \otimes L$, and therefore $\det(b_{pi})_{p,i=1}^n \in \mathbb{K}^*$. Hence, by Lemma 1 there exists a basic subalgebra \hat{L} of $W_n(\mathbb{K})$ such that \hat{L} is isomorphic to the Lie algebra L . \square

Now we can prove the main theorem of this paper.

Theorem 2. *Let L be any Lie algebra L over \mathbb{K} of dimension $n \geq 1$. Then there exists a basic subalgebra \bar{L} of $W_n(\mathbb{K})$ such that \bar{L} is isomorphic to L .*

Proof. By Proposition 2 we may assume that L is not solvable. Let $S = S(L)$ be the solvable radical of L , and $L = L_0 \ltimes S$ be the Levi decomposition of L , where L_0 is a semisimple subalgebra of L . Let $H_0 \subseteq L_0$ be a Cartan subalgebra of L_0 and let $L = N_- \oplus H_0 \oplus N_+$ be the corresponding triangular decomposition for some choice of simple roots. Denote by B_0 the Borel subalgebra $B_0 = H_0 + N_+$ of L_0 . Then the subalgebra $L_1 = S + B_0$ is solvable, therefore by Proposition 2 there exists a basic subalgebra of $W_k(\mathbb{K})$, which is isomorphic to \bar{L}_1 , where $k = \dim L_1$. Since $L = L_1 + N_-$, $L_1 \cap N_- = \{0\}$, and the subalgebra N_- acts nilpotently (by multiplication) on the L_0 -module L (see Lemma 3), there exists by Lemma 4 a basic subalgebra \bar{L} of $W_n(\mathbb{K})$ such that \bar{L} is isomorphic to the Lie algebra L . \square

Example 2. Let $L = sl_2(\mathbb{K})$ and $\{E, H, F\}$ be its standard basis over \mathbb{K} . We shall construct a basic subalgebra of $W_3(\mathbb{K})$, which is isomorphic to L . As the Cartan subalgebra $\langle H \rangle$ of L is one-dimensional, the basic subalgebra $\langle -\frac{\partial}{\partial x_1} \rangle$ of $W_1(\mathbb{K})$ is isomorphic to $\langle H \rangle$. To the element $H \in L$ corresponds the element $1 \otimes H$ in the Lie algebra \hat{L} (see the Remark 1). The subalgebra N_+ has obviously generators E and $[H, E] = 2E$. Put $w = x_2 \otimes E$. Therefore, we have

$$e^{\text{ad}(x_2 \otimes E)} \left(\frac{\partial}{\partial x_1} - 1 \otimes H \right) = \frac{\partial}{\partial x_1} + 2x_2 \otimes E - 1 \otimes H$$

$$e^{\text{ad } x_2 \otimes E} \left(\frac{\partial}{\partial x_2} \right) = \frac{\partial}{\partial x_2} - 1 \otimes E.$$

Further, N_- has generators F and $[H, F] = -2F$, $[E, F] = H$. Analogously we obtain

$$\begin{aligned} & e^{\text{ad}(x_3 \otimes F)} \left(\frac{\partial}{\partial x_1} + 2x_2 \otimes E - 1 \otimes H \right) = \\ &= \frac{\partial}{\partial x_1} + 2x_2 \otimes E - (1 + 2x_2 x_3) \otimes H - (2x_3 + 2x_2 x_3^2) \otimes F \\ & e^{\text{ad } x_3 \otimes F} \left(\frac{\partial}{\partial x_2} - 1 \otimes E \right) = \frac{\partial}{\partial x_2} - 1 \otimes E + x_3 \otimes H + x_3^2 \otimes F \\ & e^{\text{ad } x_3 \otimes F} \left(\frac{\partial}{\partial x_3} \right) = \frac{\partial}{\partial x_3} - 1 \otimes F. \end{aligned}$$

Then, using the inverse matrix to the matrix of these elements, we get a basis of $W_3(\mathbb{K})$. The linear span of this basis over \mathbb{K} is isomorphic to the Lie algebra $L = sl_2(\mathbb{K})$:

$$\begin{aligned} E &= -x_3 \frac{\partial}{\partial x_1} + (1 + 2x_2 x_3) \frac{\partial}{\partial x_2} - x_3^2 \frac{\partial}{\partial x_3} \\ H &= \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}, \quad F = \frac{\partial}{\partial x_3}. \end{aligned}$$

Remark 2. Since the set of all linear homogeneous derivations of $W_n(\mathbb{K})$ form a Lie algebra, which is isomorphic to $gl_n(\mathbb{K})$, there are embeddings of any n -dimensional Lie algebra L without center into $W_n(\mathbb{K})$. But for the image \bar{L} of L by such an embedding in $W_n(\mathbb{K})$ and a basis $\{D_1, \dots, D_n\}$ of \bar{L} such that $D_i = \sum_{j=1}^n f_{ij} \frac{\partial}{\partial x_j}$, the determinant $\det(f_{ij})$ is not a constant. Therefore \bar{L} is not a basic Lie subalgebra of $W_n(\mathbb{K})$.

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